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# ON THE NON-INJECTIVE COMPONENT AS GALOIS MODULE OF GENERALIZED JACOBIANS

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**Abstract**

Let $\ell$ be a prime number and let $L/K$ be a finite Galois $\ell$-extension of function fields of one variable with field of constants $k$, an algebraically closed field of characteristic $p \neq \ell$. In this paper, we obtain two explicit characterizations of the non-injective component of the $\ell$-part of the generalized Jacobian $\mathscr{C}_{0\mathfrak{N}}(\ell)$, where the modulus $\mathfrak{N}$ in $L$ is induced by a modulus $\mathfrak{M}$ in $K$, which contains in its support all the prime divisors of $K$ ramified in $L$. We find explicitly the decomposition of the dual of the $\ell$-part of the generalized Jacobian $\mathfrak{X}({}_{\ell}\mathscr{C}_{0\mathfrak{N}})$ as direct sum of indecomposable $\mathbb{F}_\ell[G]$-modules. We determine an exact sequence of $\mathbb{F}_\ell[G]$-modules that characterizes implicitly the $\ell$-part of the usual Jacobian $\mathscr{C}_{0L}(\ell)$ in the general case.

## 1. Introduction

Let $k$ be an algebraically closed field of characteristic $p \geq 0$, $\ell$ be a prime number, $K/k$ be an algebraic function field of one variable with field of constants $k$, and $L/K$ be a finite Galois $\ell$-extension of function fields with Galois group $\mathrm{Gal}(L/K) = G$. The group $G$ acts naturally on the $\ell$-torsion of the Jacobian variety $\mathbb{J}_L$ associated to the function field $L/k$. Hence, $G$ acts by restriction on ${}_{\ell^m}\mathbb{J}_L$, the group of points of $\mathbb{J}_L$ of order dividing $\ell^m$. Then, the direct limit $\mathbb{J}_L(\ell) := \varinjlim_{m} {}_{\ell^m}\mathbb{J}_L = \bigcup_{m=1}^{\infty} {}_{\ell^m}\mathbb{J}_L$ has a $\mathbb{Z}_\ell[G]$-module structure, where $\mathbb{Z}_\ell$ denotes the ring of $\ell$-adic integers and $\mathbb{Z}_\ell[G]$ denotes the group ring over $\mathbb{Z}_\ell$. We have that $\mathbb{J}_L(\ell)$ is naturally $G$-isomorphic to $\mathscr{C}_{0L}(\ell)$, the Sylow $\ell$-subgroup of the group $\mathscr{C}_{0L}$ of divisor classes of degree 0 of $L$. It is well known that, for $\ell \neq p$, $\mathscr{C}_{0L}(\ell) \cong R^{2g_L}$ as groups, where $g_L$ denotes the genus of $L$, $R := \frac{\mathbb{Q}_\ell}{\mathbb{Z}_\ell}$, and $\mathbb{Q}_\ell$ denotes the field of $\ell$-adic numbers.

The basic tool used with success in the study of the Galois module structure of the usual Jacobian $\mathscr{C}_{0L}(\ell)$, i.e., in the obtention of the

decomposition of $\mathscr{C}_{0L}(\ell)$ as direct sum of indecomposable $\mathbb{Z}_\ell[G]$-modules, in both cases $\ell = p$ and $\ell \neq p$, has turned out to be the consideration of the generalized Jacobian variety $\mathscr{C}_{\mathfrak{N}}$, where the modulus $\mathfrak{N}$ in $L$ is induced from a modulus $\mathfrak{M}$ in $K$, which contains in its support all the primes ramified in $L$, and the exact sequence

$$0 \longrightarrow \mathfrak{R} \longrightarrow \mathscr{C}_{0\mathfrak{N}}(\ell) \longrightarrow \mathscr{C}_{0L}(\ell) \longrightarrow 0, \tag{1}$$

where $\mathscr{C}_{0\mathfrak{N}}(\ell)$ is the $\ell$-torsion of $\mathscr{C}_{0\mathfrak{N}}$ and $\mathfrak{R}$ is the kernel of the natural map, which was characterized as $\mathbb{Z}_\ell[G]$-module by Villa and Madan (see [17], Theorem 1, page 257).

When $\ell = p$, we have that the generalized Jacobian $\mathscr{C}_{0\mathfrak{N}}(\ell)$ is an injective $\mathbb{Z}_\ell[G]$-module, that is, $\mathscr{C}_{0\mathfrak{N}}(\ell) \cong R[G]^u$, for some $u \geq 0$ (see [16], Proposition 8).

In the case $\ell \neq p$, $\mathscr{C}_{0\mathfrak{N}}(\ell)$ is always non-injective as $\mathbb{Z}_\ell[G]$-module (see [18], Theorem 6). In fact, one has that $\mathscr{C}_{0\mathfrak{N}}(\ell) \cong R[G]^u \oplus S$ with $S$ an indecomposable $\mathbb{Z}_\ell[G]$-module, which is isomorphic to $R^s$ as groups, where $s = |G|(d-1)+1$, $d$ denotes the minimum number of generators of $G$ and $|G|$ denotes the order of $G$.

The use of the dual of Heller's loop operator $\Omega^{\#}$ has been very effective in the study of the non-injective component of ${}_\ell\mathscr{C}_{0L}$, the $\ell$-part of $\mathscr{C}_{0L}(\ell)$. More explicitly, we want to establish an exact sequence of $\mathbb{F}_\ell[G]$-modules

$$0 \longrightarrow M \longrightarrow \mathbb{F}_\ell[G]^\beta \longrightarrow {}_\ell\mathscr{C}_{0L} \longrightarrow 0, \tag{2}$$

where $\mathbb{F}_\ell$ denotes the finite field with $\ell$ elements.

To relate the $\ell$-parts of sequences (1) and (2), it is very important to know explicitly the structure as $\mathbb{F}_\ell[G]$-module of ${}_\ell\mathscr{C}_{0\mathfrak{N}}$, the $\ell$-part of the generalized Jacobian $\mathscr{C}_{0\mathfrak{N}}(\ell)$.

In particular, since ${}_\ell\mathscr{C}_{0\mathfrak{N}}$ is the subgroup of elements of order dividing $\ell$ of $\mathscr{C}_{0\mathfrak{N}}(\ell)$, we have that ${}_\ell\mathscr{C}_{0\mathfrak{N}}$ is an $\mathbb{F}_\ell[G]$-module and

$$ {}_\ell\mathscr{C}_{0\mathfrak{N}} \cong \mathbb{F}_\ell[G]^u \oplus {}_\ell S, $$

where we denote by ${}_\ell S$, the $\ell$-part of the $\mathbb{Z}_\ell[G]$-module $S$.

In this paper, we obtain two explicit characterizations of the $\mathbb{F}_\ell[G]$-module ${}_\ell S$, Theorem 3.8, and Proposition 4.6. For $L/K$, any finite Galois unramified $\ell$-extension, in Section 5, we obtain explicitly the decomposition of ${}_\ell\mathscr{C}_{0L}$ as direct sum of indecomposable $\mathbb{F}_\ell[G]$-modules, this is (36). We determine explicitly the Galois module structure of the dual of the $\ell$-part of the generalized Jacobian $\mathfrak{X}({}_\ell\mathscr{C}_{0\mathfrak{N}})$, that is, we obtain explicitly the decomposition of $\mathfrak{X}({}_\ell\mathscr{C}_{0\mathfrak{N}})$ as a direct sum of indecomposable $\mathbb{F}_\ell[G]$-modules, this structure is obtained in Theorem 5.3. In Section 2, we collect basic results that we use along the paper. In Sections 3 and 4, we give the proofs of Theorem 3.8 and Proposition 4.6, respectively.

In Section 5, we obtain an exact sequence of $\mathbb{F}_\ell[G]$-modules that implicitly characterizes the Galois module structure of the $\ell$-part of usual Jacobian $\mathscr{C}_{0L}(\ell)$, i.e., we determine the value of $\beta$ appearing in (2), for an arbitrary Galois $\ell$-extension $L/K$.

## 2. Notations and Auxiliary Results

Now, we introduce the objects that we will be working along with the paper. $L/K$ denotes a finite Galois $\ell$-extension of function fields of order $\ell^n$ with Galois group $G = \mathrm{Gal}(L/K)$ and field of constants $k$, an algebraically closed field of characteristic $p \neq \ell$.

Let

$$ \mathscr{P} = \{\mathscr{P}_1, \mathscr{P}_2, \ldots, \mathscr{P}_t\}, $$

and

$$\widehat{\mathscr{P}} = \left\{ \mathscr{Q}_j^{(i)} \middle| i \in \{1, \ldots, t\},\ j \in \{1, \ldots, \ell^{n-n_i}\} \right\},$$

where $\mathscr{P}$ consists of all the different prime divisors of $K$ ramified in $L$, $\widehat{\mathscr{P}}$ is the set of prime divisors $\mathscr{Q}_j^{(i)}$ of $L$ such that $\mathscr{Q}_j^{(i)}$ divides the prime divisor $\mathscr{R}_i$, for $1 \leq j \leq \ell^{n-n_i}$, and $\ell^{n_i}$ denotes the ramification index of the prime divisor $\mathscr{R}_i$.

Let $\mathfrak{M}$ be the modulus in $K$ defined by:

$$\mathfrak{M} = \prod_{i=1}^{t} \mathscr{R}_i.$$

Let $\mathfrak{N}$ be the modulus in $L$ induced by $\mathfrak{M}$ (i.e., $\mathfrak{N}$ is the conorm of $\mathfrak{M}$), given by:

$$\mathfrak{N} = \prod_{\mathscr{Q} \in \widehat{\mathscr{P}}} \mathscr{Q}.$$

We use the following notation:

- $\mathbb{P}_L$ is the set of prime divisors of $L$,
- $\mathscr{D}_{\mathfrak{N}}$ is the group of divisors of $L$ relatively prime to $\mathfrak{N}$,
- $\mathscr{D}_{0\mathfrak{N}}$ is the group of divisors of degree zero relatively prime to $\mathfrak{N}$,
- $P_{\mathfrak{N}}$ is the group of principal divisors $(\alpha)$ such that $\alpha \equiv 1 \bmod \mathfrak{N}$,
- $\mathscr{C}_{0\mathfrak{N}} := \dfrac{\mathscr{D}_{0\mathfrak{N}}}{P_{\mathfrak{N}}}$ is the group of classes of divisors of degree 0 associated to $\mathfrak{N}$,
- $L_{\mathfrak{N}}$ is the group of $\alpha \in L$ such that $(\alpha)$ is relatively prime to $\mathfrak{N}$, and
- $L_{\mathfrak{N}^{(1)}}$ is the group of $\alpha \in L_{\mathfrak{N}}$ such that $\alpha \equiv 1 \bmod \mathfrak{N}$.

We denote by $\mathscr{C}_{0\mathfrak{N}}(\ell) \cong \mathbb{J}_{\mathfrak{N}}(\ell)$, the Sylow $\ell$-subgroup of $\mathscr{C}_{0\mathfrak{N}}$, and we call it *generalized Jacobian*.

For any $G$-module $A$, the $i$-th Tate cohomology group $\widehat{H}^i(G, A)$ with $i \in \mathbb{Z}$, is denoted by $H^i(G, A)$. The trivial group is denoted by 0, whether its structure is additive or multiplicative. Also, we denote the elements of $A$ fixed by the action of $G$, by $A^G := \{m \in A | gm = m \forall g \in G\}$ and by $N_G = N$ the norm (or trace) map, that is, if $m \in A$, then $N(m) = \prod_{g \in G} gm \ (\text{or} \sum_{g \in G} gm)$. Sometimes, we will use the additive notation $N(A)$ and some other the multiplicative notation $A^N$ for the norm of $A$. On the other hand, ${}_N A$ denotes the kernel of $N$ acting on $A$, $I_G A$ denotes the module generated by $\langle gm - m | g \in G, m \in A \rangle$ and $I_G = \langle g - 1 | g \in G \rangle \subseteq \mathbb{Z}_\ell[G]$. We denote by $C_m$ the cyclic group with $m$ elements.

We remark that some results of this section are very well-known, however, we decided to include them, some with their proofs, to make the reading of the paper easier.

One of the main results, very frequently used during the present work is the following result of which we present its proof.

**Lemma 2.1** (Schanuel's lemma for projective modules)**.** *Let $P_1$ and $P_2$ be projective $A$-modules, where $A$ is a commutative ring with identity. If we have two exact sequences of $A$-modules*:

$$0 \to B_1 \to P_1 \to X \to 0,$$

$$0 \to B_2 \to P_2 \to X \to 0,$$

*then $P_1 \oplus B_2 \cong P_2 \oplus B_1$.*

**Proof.** Let $\varphi : P_1 \to X$ and $\varphi' : P_2 \to X$ denote the epimorphisms of the exact sequences. Let

$$M = \{(p, q) \in P_1 \oplus P_2 | \varphi(p) = \varphi'(q)\}.$$

If $\pi : M \to P_1$ is the projection on the first coordinate of $M$, then $\pi$ is an epimorphism. In fact, since $\varphi'$ is epimorphism, for every $p \in P_1$, there exists $q \in P_2$ such that $\varphi(p) = \varphi'(q)$, then $(p, q) \in M$ and $\pi(p, q) = p$. On the other hand,

$$\ker(\pi) = \{(0, q) | \varphi'(q) = 0\}$$

$$\cong \ker(\varphi') \cong B_2.$$

Therefore, we have the exact sequence

$$0 \to B_2 \to M \to P_1 \to 0.$$

Since $P_1$ is projective this exact sequence splits, thus $M \cong B_2 \oplus P_1$. Similarly, we have an epimorphism $\pi_1 : M \to P_2$ and using the same previous argument, we obtain an exact sequence $0 \to B_1 \to M \to P_2 \to 0$. Therefore, $M \cong B_1 \oplus P_2$, proving the result. □

**Remark 2.2.** We have a dual of Schanuel's lemma. That is, Schanuel's lemma for injective modules.

Let $P_1$ and $P_2$ be two injective $A$-modules, where $A$ is a commutative ring with identity. Let us assume that we have two exact sequences of $A$-modules:

$$0 \to X \to P_1 \to B_1 \to 0,$$

$$0 \to X \to P_2 \to B_2 \to 0,$$

then $P_1 \oplus B_2 \cong P_2 \oplus B_1$.

**Remark 2.3.** If $M$ is an $\mathbb{F}_\ell[G]$-module, then $M$ is an injective $\mathbb{F}_\ell[G]$-module, if and only if $M$ is a projective $\mathbb{F}_\ell[G]$-module. Furthermore, when $M$ is an $\mathbb{F}_\ell[G]$-module, Lemma 2.1, and Remark 2.2 are equivalent.

For any $\mathbb{Z}_\ell[G]$-module $M$, we define $\mathfrak{X}(M) := \mathrm{Hom}_{\mathbb{Z}_\ell}(M, R)$, *the Pontrjagin's dual of* $M$. We have what $\mathfrak{X}(M)$ has a $\mathbb{Z}_\ell[G]$-module structure given by:

if $f \in \mathfrak{X}(M)$, $g \in G$, and $x \in M$, then $(g \cdot f)(x) = f(g^{-1} \cdot x)$.

A fundamental result on the Pontrjagin's dual is the following proposition. The proof can be found in [8], page 84.

**Proposition 2.4** (Pontrjagin-Van Kampen)**.** *Let $G$ be a finite $\ell$-group and let $M$ be a $\mathbb{Z}_\ell[G]$-module such that $M$ as group is locally compact in the compact-open topology. Then, $\mathfrak{X}(\mathfrak{X}(M)) \cong M$ as $\mathbb{Z}_\ell[G]$-modules.* □

**Lemma 2.5.** *Let $G$ be a finite $\ell$-group and let $H$ be a subgroup of $G$. Then*

(i) *$\mathfrak{X}(\mathbb{Z}_\ell) \cong R$ and $\mathfrak{X}(R) \cong \mathbb{Z}_\ell$ as $\mathbb{Z}_\ell[G]$-modules.*

(ii) *$\mathfrak{X}(\mathbb{Z}_\ell[G/H]) \cong R[G/H]$ and $\mathfrak{X}(R[G/H]) \cong \mathbb{Z}_\ell[G/H]$ as $\mathbb{Z}_\ell[G]$-modules.*

(iii) *If $0 \to A \to B \to C \to 0$ is an exact sequence of $\mathbb{Z}_\ell[G]$-modules, then $0 \to \mathfrak{X}(C) \to \mathfrak{X}(B) \to \mathfrak{X}(A) \to 0$ is an exact sequence of $\mathbb{Z}_\ell[G]$-modules.*

**Proof.** We will only show the first isomorphisms of (i) and (ii), the others are an immediate consequence of Proposition 2.4. If $\alpha \in \mathfrak{X}(\mathbb{Z}_\ell) = \mathrm{Hom}_{\mathbb{Z}_\ell}(\mathbb{Z}_\ell, R)$, let $\theta : \mathfrak{X}(\mathbb{Z}_\ell) \to R$ be given by $\theta(\alpha) = \alpha(1)$. It is easy to see that $\theta$ is an isomorphism, which shows (i).

Let $\varphi \in \mathrm{Hom}_{\mathbb{Z}_\ell}(\mathbb{Z}_\ell[G/H], R)$. Then $G$ acts on $\varphi$ by $(g \circ \varphi)(a) = \varphi(g^{-1}a)$, with $g \in G$, and $a \in \mathbb{Z}_\ell[G/H]$. Let $\theta : \mathfrak{X}(\mathbb{Z}_\ell[G/H]) \to R[G/H]$ be given by $\theta(\varphi) = \sum_{\sigma \in G/H} \varphi(\sigma)\sigma$. It is easy to see that $\theta$ is a $\mathbb{Z}_\ell[G]$-isomorphism, this shows (ii).

On the other hand, since $R[G]$ is an injective $\mathbb{Z}_\ell[G]$-module and $0 \to A \to B \to C \to 0$ is an exact sequence of $\mathbb{Z}_\ell[G]$-modules, by Theorem 8.4 of [2], page 36, we obtain the following exact sequence of $\mathbb{Z}_\ell[G]$-modules

$$0 \to \mathrm{Hom}_{\mathbb{Z}_\ell[G]}(C, R[G]) \to \mathrm{Hom}_{\mathbb{Z}_\ell[G]}(B, R[G]) \to \mathrm{Hom}_{\mathbb{Z}_\ell[G]}(A, R[G]) \to 0.$$

Finally, since $\mathrm{Hom}_{\mathbb{Z}_\ell[G]}(M, R[G]) \cong (\mathrm{Hom}_{\mathbb{Z}_\ell}(M, R[G]))^G \cong \mathrm{Hom}_{\mathbb{Z}_\ell}(M, R)$, as $\mathbb{Z}_\ell[G]$-modules, we obtain (iii). □

If $P$ is an $\mathbb{F}_\ell[G]$-module, $P^* := \mathrm{Hom}_{\mathbb{F}_\ell}(P, \mathbb{F}_\ell)$ denotes the dual of $P$ as $\mathbb{F}_\ell$-vector space. We have what $P^*$ has an $\mathbb{F}_\ell[G]$-module structure given by:

$$\text{if } f \in P^*,\ g \in G, \text{ and } x \in P, \text{ then } (g \cdot f)(x) = f(g^{-1} \cdot x).$$

**Remark 2.6.** Since $\mathbb{F}_\ell \cong_\ell R \subseteq R$, we have that for any $\mathbb{F}_\ell[G]$-module $P$, $P^* \cong \mathfrak{X}(P)$ as $\mathbb{F}_\ell[G]$-modules.

Let $\psi : \mathbb{F}_\ell[G] \to \mathbb{F}_\ell$ be the *augmentation* $\mathbb{F}_\ell[G]$-*epimorphism* given by:

$$\psi\left(\sum_{g \in G} a_g g\right) := \sum_{g \in G} a_g.$$

The $\mathbb{F}_\ell[G]$-module, ${}_\ell I_G := \ker(\psi) = \{\sum_{g \in G} a_g g \in \mathbb{F}_\ell[G] \,\big|\, \sum_{g \in G} a_g = 0\}$ is called the *augmentation ideal* of $\mathbb{F}_\ell[G]$.

**Lemma 2.7.** *Let $G$ be a finite $\ell$-group and let $P$ be an $\mathbb{F}_\ell[G]$-module. Then*

(i) $\dim_{\mathbb{F}_\ell}(P) = \dim_{\mathbb{F}_\ell}(P^*)$ *and* $P \cong (P^*)^*$ *as* $\mathbb{F}_\ell[G]$-*modules.*

(ii) $(\mathbb{F}_\ell[G])^* \cong \mathbb{F}_\ell[G]$ *as* $\mathbb{F}_\ell[G]$-*modules.*

(iii) $\left(\frac{\mathbb{F}_\ell[G]}{\mathbb{F}_\ell}\right)^* \cong {}_\ell I_G$ *as* $\mathbb{F}_\ell[G]$-*modules.*

(iv) *If* $0 \to A \to B \to C \to 0$ *is an exact sequence of* $\mathbb{F}_\ell[G]$-*modules, then* $0 \to C^* \to B^* \to A^* \to 0$ *is an exact sequence of* $\mathbb{F}_\ell[G]$-*modules.*

**Proof.** (i) and (iv), are similar to Lemma 2.5. For (ii) and (iii), see [16], Lemma 7, page 344. □

Let $M$ be a $\mathbb{Z}_\ell[G]$-module and $0 \to M \to Y \to P \to 0$ be any exact sequence with $Y$ an injective $\mathbb{Z}_\ell[G]$-module, we write $P = P^{(1)} \oplus P^{(0)}$, where $P^{(1)}$ is an injective $\mathbb{Z}_\ell[G]$-module and $P^{(0)}$ has no $\mathbb{Z}_\ell[G]$-injective components. Then, $\Omega^{\#}(M) := P^{(0)}$ is *the dual of Heller's loop operator of M*. The module $\Omega^{\#}(M)$ is unique up to isomorphism. Note that $\Omega^{\#}$ is well defined, since the Krull-Schmidt-Azumaya's theorem (see [1], (6.12), page 128) holds for $\mathbb{Z}_\ell[G]$-modules.

We have a concept dual to $\Omega^{\#}$. Let $Y$ be a projective $\mathbb{Z}_\ell[G]$-module and write $M = M^{(1)} \oplus M^{(0)}$, where $M^{(1)}$ is a projective $\mathbb{Z}_\ell[G]$-module and $M^{(0)}$ has no $\mathbb{Z}_\ell[G]$-projective components. Then, $\Omega(P) := M^{(0)}$ is *the Heller's loop operator of P*. The module $M^{(0)}$ is unique up to isomorphism.

We have (see [17], Proposition 4, page 258).

**Proposition 2.8.** *Let G be a finite $\ell$-group and let H be a subgroup of G. Then*

(i) $R[G/H]$ *and* $\dfrac{R[G]}{R[G/H]}$ *are indecomposable* $\mathbb{Z}_\ell[G]$*-modules.*

(ii) $\Omega^{\#}(R[G/H]) \cong \dfrac{R[G]}{R[G/H]}$ *as* $\mathbb{Z}_\ell[G]$*-modules.*

(iii) *If* $M_1$ *and* $M_2$ *are* $\mathbb{Z}_\ell[G]$*-modules, then* $\Omega^{\#}(M_1 \oplus M_2) \cong \Omega^{\#}(M_1) \oplus \Omega^{\#}(M_2)$. □

**Proposition 2.9.** *Let G be a finite $\ell$-group and let M be a* $\mathbb{Z}_\ell[G]$*-module such that* $\mathfrak{X}(M)$ *is a finitely generated* $\mathbb{Z}_\ell[G]$*-module. Then,*

$$\mathfrak{X}(\Omega^{\#}(M)) \cong \Omega(\mathfrak{X}(M)).$$

**Proof.** See [5], Chapter 7, Theorem 5.1, page 348. □

The following results are mainly used for the study of the injective component of a $\mathbb{Z}_\ell[G]$-module.

**Theorem 2.10** ([14], Valentini)**.** *Let $F$ be an algebraically closed field of characteristic $\ell$, $G$ be a finite $\ell$-group, $M$ be a finitely generated $F[G]$-module, and let $N$ denote the norm map. If $n = \dim_F N(M)$, then $M \cong F[G]^n \oplus P$, where $F[G]$ is not a component of $P$.* □

This result still holds removing the hypothesis that $F$ is an algebraically closed field. The proof is the same as the one given in [14].

Let $M$ be a $\mathbb{Z}_\ell[G]$-module such that the Pontryagin's dual $\mathfrak{X}(M)$ is finitely generated, $G$ being a finite $\ell$-group and $M$ $\mathbb{Z}_\ell$-injective and as groups, $M \cong R^{s_0}$ with $s_0 < \infty$. If ${}_\ell M$ denotes the set of elements of $M$, whose order divide $\ell$, then ${}_\ell M$ is a finitely generated $\mathbb{F}_\ell[G]$-module. We have

**Theorem 2.11** ([9], Rzedowski-Villa-Madan)**.** *Let $M$ and $G$ be given as above. If ${}_\ell M \cong \mathbb{F}_\ell[G]^n \oplus U$, where $\mathbb{F}_\ell[G]$ is not a component of $U$ and $M \cong R[G]^m \oplus V$, where $R[G]$ is not a component of $V$, then $n = m$.* □

## 3. Non-Injective Component of Generalized Jacobians

The goal in this section is to obtain an explicit characterization of the non-injective component of the $\ell$-part of generalized Jacobians ${}_\ell\mathscr{C}_{0\mathfrak{N}}$. This is Theorem 3.8.

The general $\mathbb{Z}_\ell[G]$-module structure of $\mathscr{C}_{0\mathfrak{N}}(\ell)$, that is, the decomposition of $\mathscr{C}_{0\mathfrak{N}}(\ell)$ as direct sum of indecomposable $\mathbb{Z}_\ell[G]$-modules is given by:

**Theorem 3.1** ([18], Villa-Rzedowski)**.** *If $L/K$ is a finite Galois $\ell$-extension of function fields of one variable, then the $\mathbb{Z}_\ell[G]$-module structure of the generalized Jacobian $\mathbb{J}_{\mathfrak{N}}(\ell)$ is given by*

$$\mathbb{J}_{\mathfrak{N}}(\ell) \cong R[G]^{2g_K+t-1-d} \bigoplus S, \tag{3}$$

*where $t$ is the number of prime divisors of $K$ ramified in $L$, $S$ an indecomposable $\mathbb{Z}_\ell[G]$-module such that, as groups, $S \cong R^s$ with $s = |G|(d-1)+1$ and $d$ is the minimum number of generators of the Galois group $G$.* □

Furthermore, in [18], page 46, it was obtained:

$$H^i(G, S) \cong H^{i-1}(G, \mathbb{Z}), \text{ for all } i \in \mathbb{Z}. \tag{4}$$

In particular, for the $\ell$-part of $\mathbb{J}_{\mathfrak{N}}(\ell)$, we obtain from (3),

$${}_\ell\mathbb{J}_{\mathfrak{N}} \cong {}_\ell\mathscr{C}_{0\mathfrak{N}} \cong \mathbb{F}_\ell[G]^{2g_K+t-1-d} \bigoplus ({}_\ell S). \tag{5}$$

In general, an explicit description for the indecomposable $\mathbb{Z}_\ell[G]$-module $S$ (non-injective component of $\mathscr{C}_{0\mathfrak{N}}(\ell)$) is not known. However, we present a conjecture on the explicit description of $S$, based in a characterization of ${}_\ell S$, that we obtain in Theorem 3.8.

Let $M$ be a $\mathbb{Z}_\ell[G]$-module, that is, $\mathbb{Z}_\ell$-divisible and such that $\mathfrak{X}(M)$ is finitely generated, that is, as groups, $M \cong R^{m_0}$, for some $m_0 \in \mathbb{N} \cup \{0\}$. We have

$$0 \to {}_\ell M \to M \xrightarrow{\ell} M \to 0, \tag{6}$$

is an exact sequence of $\mathbb{Z}_\ell[G]$-modules, where $\ell$ denotes the homomorphism multiplication by $\ell$ on $M$ and ${}_\ell M := \{x \in M | x$ is of order dividing $\ell\}$.

From (6), we obtain the exact sequence of cohomology groups

$$\cdots \to H^{i-1}(G, M) \xrightarrow{\ell} H^{i-1}(G, M) \to H^i(G, {}_\ell M) \to \tag{7}$$

$$\to H^i(G, M) \xrightarrow{\ell} H^i(G, {}_\ell M) \to \cdots.$$

From (7), it follows that

$$H^i(G,\ {}_\ell M) \cong C_\ell^{\alpha_{i-1}(M)+\alpha_i(M)}, \tag{8}$$

where

$$\alpha_i(M) = \dim_{\mathbb{F}_\ell} \frac{H^i(G,\ M)}{\ell H^i(G,\ M)} = \dim_{\mathbb{F}_\ell} {}_\ell H^i(G,\ M).$$

Then

$$H^i(G,\ {}_\ell S) \cong C_\ell^{\alpha_{i-1}(S)+\alpha_i(S)}. \tag{9}$$

The cohomology of $\mathbb{Z}$, (see [21], Corollary 4.4.7), is given by

$$H^i(G,\ \mathbb{Z}) \cong H^{-i}(G,\ \mathbb{Z}), \text{ for all } i \in \mathbb{Z}, \tag{10}$$

$$H^1(G,\ \mathbb{Z}) \cong H^{-1}(G,\ \mathbb{Z}) \cong \{0\}, \tag{11}$$

$$H^0(G,\ \mathbb{Z}) \cong C_{|G|}, \tag{12}$$

$$\text{and } H^{-2}(G,\ \mathbb{Z}) \cong H^2(G,\ \mathbb{Z}) \cong G\,/\,G', \tag{13}$$

where $G'$ denotes the commutator subgroup of $G$.

In [13], Chapters 1, 4, Sections 4.3 and 4.4, it is proven that

$$\alpha_2(\mathbb{Z}) = d \quad \text{and} \quad \alpha_3(\mathbb{Z}) = r - d, \tag{14}$$

where $d$ is the minimum number of generators of $G$ and $r$ is the number of relations of $G$.

**Proposition 3.2.** *If $S$ is the $\mathbb{Z}_\ell[G]$-module appearing in Theorem* 3.1, *then*

$$\dim_{\mathbb{F}_\ell}({}_\ell S^G) = d.$$

**Proof.** From (9), (4), and (14), we have

$$H^0(G,\ {}_\ell S) \cong C_\ell^{\alpha_0(S)+\alpha_{-1}(S)} \cong C_\ell^{0+d} \cong C_\ell^d.$$

From Theorems 3.1 and 2.10, we obtain that $\dim_{\mathbb{F}_\ell} N({}_\ell S) = 0$. Since $H^0(G, {}_\ell S) \cong \frac{{}_\ell S^G}{N({}_\ell S)}$, we have $H^0(G, {}_\ell S) \cong {}_\ell S^G \cong C_\ell^d$. The result follows. □

**Lemma 3.3.** *If* $\dim_{\mathbb{F}_\ell}({}_\ell S^G) = c$ *and* $c'$ *is the minimum natural number such that* $g : S \to R[G]^{c'}$ *is a* $\mathbb{Z}_\ell[G]$*-monomorphism, then* $c = c'$.

**Proof.** Since $g : S \to R[G]^{c'}$ is a $\mathbb{Z}_\ell[G]$-monomorphism, we have the exact sequence

$$0 \to S \to R[G]^{c'} \to \frac{R[G]^{c'}}{S} \to 0.$$

Considering cohomology groups, we obtain the exact sequence

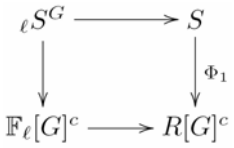
$$0 \to S^G \to R^{c'} \to \left(\frac{R[G]^{c'}}{S}\right)^G \to \cdots.$$

Taking $\ell$-parts, we have the exact sequence

$$0 \to {}_\ell S^G \to \mathbb{F}_\ell^{c'} \to \ell\left(\frac{R[G]^{c'}}{S}\right)^G \to \cdots.$$

Therefore $\sigma : {}_\ell S^G \to \mathbb{F}_\ell^{c'}$ is a monomorphism, therefore, $\dim_{\mathbb{F}_\ell}({}_\ell S^G) \leq c'$.

On the other hand, since $\dim_{\mathbb{F}_\ell}({}_\ell S^G) = c$, it follows that ${}_\ell S^G \cong \mathbb{F}_\ell^c$, as groups. Furthermore, since $R[G]^c$ is an injective $\mathbb{Z}_\ell[G]$-module, it is obtained the following commutative diagram



where $\Phi_1$ is a $\mathbb{Z}_\ell[G]$-monomorphism, thus $c' \leq c$. This shows the result.

□

**Corollary 3.4.** *Let $A$ be a $\mathbb{Z}_\ell[G]$-module and let $e$ be the minimum natural number such that there exists a $\mathbb{Z}_\ell[G]$-monomorphism $g : A \to R[G]^e$. Then $e = \dim_{\mathbb{F}_\ell}({}_\ell A^G)$.*

**Proof.** Let $\beta = \dim_{\mathbb{F}_\ell}({}_\ell A^G)$. Using Lemma 3.3, we have $\beta = e$. □

Let $\psi : R[G] \to R$ be the *augmentation $\mathbb{F}_\ell[G]$-epimorphism* given by:

$$\psi\left(\sum_{g\in G} a_g g\right) := \sum_{g\in G} a_g.$$

The $R[G]$-module, $I_G := \ker(\psi) = \{\sum_{g\in G} a_g g \in R[G] \,|\, \sum_{g\in G} a_g = 0\}$ is called the *augmentation module of $R[G]$*.

**Theorem 3.5.** *If $S$ is the $\mathbb{Z}_\ell[G]$-module appearing in Theorem* 3.1, *we have*

$$0 \to S \to R[G]^d \to I_G \to 0, \tag{15}$$

*is an exact sequence of $\mathbb{Z}_\ell[G]$-modules, where as groups, we have $R[G]^d \cong R^{|G|d}$, $I_G \cong R^{|G|-1}$, and $S \cong R^{|G|(d-1)+1}$.*

**Proof.** From Proposition 3.2 and Lemma 3.3, we obtain the exact sequence of $\mathbb{Z}_\ell[G]$-modules

$$0 \to S \to R[G]^d \to M \to 0. \tag{16}$$

The result will be proved, if $M \cong I_G$ as $\mathbb{Z}_\ell[G]$-modules.

Since $R[G]$ is cohomologically trivial, we have $H^i(G, M) \cong H^{i+1}(G, S)$. In particular, from (4) and (12), we obtain

$$H^0(G, M) \cong H^1(G, S) \cong H^0(G, \mathbb{Z}) \cong C_{|G|}.$$

From (16), we obtain the exact sequence of cohomology groups

$$0 \to S^G \to (R[G]^d)^G \to M^G \to H^1(G, S) \to 0,$$

where $(R[G]^d)^G \cong R^d$ and $S^G \cong R^d$. Therefore, $M^G \cong H^1(G, S)$.

Since $C_{|G|} \cong H^1(G, S) \cong H^0(G, M) \cong \frac{M^G}{N(M)}$, we have $N(M) = 0$, i.e., $N(M) \subseteq_N (R[G]) = \ker(N(R[G]))$. On the other hand, if $N : R[G] \to R[G]$ is the $\mathbb{Z}_\ell[G]$-homomorphism given by $N(\sum_{g\in G} a_g g) = \sum_{\tau\in G} \sum_{g\in G} a_g g$, then ${}_N(R[G]) = I_G$. Therefore $M \subseteq I_G$. From the above, one has the exact sequence of $\mathbb{Z}_\ell[G]$-modules

$$0 \to M \to I_G \to \frac{I_G}{M} \to 0.$$

Since $H^2(G, S) \cong H^1(G, M) \cong \{0\}$, we obtain the exact sequence of $G$-modules

$$0 \to M^G \to I_G^G \to \left(\frac{I_G}{M}\right)^G \to 0.$$

Thus,

$$0 \to C_{|G|} \to C_{|G|} \to \left(\frac{I_G}{M}\right)^G \to 0.$$

Hence $\left(\frac{I_G}{M}\right)^G \cong \{0\}$, i.e., $\frac{I_G}{M} = \{0\}$, that is, $I_G \cong M$. □

Now, from Proposition 3.2 and Lemma 3.3, the indecomposability of $S$, and the definition of $\Omega^\#$, we obtain the exact sequence of $\mathbb{Z}_\ell[G]$-modules

$$0 \to S \to R[G]^d \to \Omega^\#(S) \to 0, \tag{17}$$

where, as groups, we have $S \cong R^{|G|(d-1)+1}$, $R[G]^d \cong R^{|G|d}$, and $\Omega^{\#}(S) \cong R^{|G|-1}$.

From (17), and using that $R[G]$ is cohomologically trivial, we obtain the exact sequence of cohomology groups

$$0 \to S^G \to (R[G]^d)^G \to (\Omega^{\#}(S))^G \to H^1(S) \to 0. \tag{18}$$

From (4), in particular, we have $H^1(G, S) \cong H^0(G, \mathbb{Z}) \cong C_{|G|}$.

Thus, $S^G \cong (R[G]^d)^G \cong R^d$ imply that $\Omega^{\#}(S)^G \cong H^1(G, S) \cong C_{|G|}$. We have proved:

**Proposition 3.6.** *With the above notations, we have* $\dim_{\mathbb{F}_\ell}({}_\ell\Omega^{\#}(S)^G) = 1$. □

**Proposition 3.7.** *With the notation as above, we have*

$$I_G \cong \Omega^{\#}(S).$$

**Proof.** We obtain the result applying Schanuel's lemma (see Remark 2.2) to the exact sequences (17) and (15). □

**Theorem 3.8.** *Let* $0 \to W \to \mathbb{F}_\ell[G]^d \to {}_\ell I_G \to 0$ *be any exact sequence of G-modules, with d the minimum number of generators of G. Then*

$${}_\ell S \cong W.$$

*More explicitly, we consider the epimorphism*

$$\vartheta : \mathbb{F}_\ell[G]^d \to {}_\ell I_G := \langle \sigma - 1 | \sigma \in G \rangle,$$

*given by*

$$\vartheta(\xi_1, \xi_2, \ldots, \xi_d) := \sum_{i=1}^{d} \xi_i(\sigma_i - 1),$$

*where* $G = \langle \sigma_1, \ldots, \sigma_d \rangle$ *and* $\vartheta$ *is induced by*:

$$\vartheta(1, 0, \ldots, 0, 0) := \sigma_1 - 1,$$

$$\vartheta(0, 1, \ldots, 0, 0) := \sigma_2 - 1,$$

$$\vdots \qquad\qquad \vdots$$

$$\vartheta(0, 0, \ldots, 0, 1) := \sigma_d - 1.$$

*Then* ${}_\ell S \cong \ker(\vartheta)$.

**Proof.** Taking $\ell$-parts in the exact sequence (15), we obtain the exact sequences

$$0 \to {}_\ell S \to \mathbb{F}_\ell[G]^d \to {}_\ell I_G \to 0,$$

$$0 \to W \to \mathbb{F}_\ell[G]^d \to {}_\ell I_G \to 0.$$

We have the statement using Schanuel's lemma (see Remark 2.3). □

Now, keeping in mind Theorem 3.8 and the cyclic case, that is, if $L/K$ is a cyclic finite $\ell$-extension, the non-injective component of the Galois module structure of the generalized Jacobian $\mathscr{C}_{0\mathfrak{M}}(\ell)$ is $S \cong R$. In this case, $d = 1$ and $S \cong R$, it is characterized by the exact sequence

$$0 \to S \cong R \to R[G] \to I_G \to 0.$$

In other words, the cylic case gives evidence for the following:

**Conjecture 3.9.** Let $I_G$ be the augmentation module and $\Theta : R[G]^d \to I_G$, the epimorphism given by

$$\Theta(\xi_1, \xi_2, \ldots, \xi_d) := \sum_{i=1}^{d} \xi_i(\sigma_i - 1),$$

where $G = \langle \sigma_1, \ldots, \sigma_d \rangle$, $d$ is the minimum number of generators of $G$ and $\Theta$ is induced by:

$$\Theta(1, 0, \ldots, 0, 0) := \sigma_1 - 1,$$

$$\Theta(0, 1, \ldots, 0, 0) := \sigma_2 - 1,$$

$$\vdots \qquad\qquad \vdots$$

$$\Theta(0, 0, \ldots, 0, 1) := \sigma_d - 1.$$

Then, $S \cong \ker(\Theta)$.

## 4. A *G*-Exact Sequence of ${}_\ell S$

The main result of this section is Theorem 4.3, which has as a consequence to establish an exact sequence of ${}_\ell S$. It allows us to obtain another characterization of ${}_\ell S$. The following result is used for its proof.

**Proposition 4.1.** *Let $A$ be a $\mathbb{F}_\ell[G]$-module,*

$\alpha = \min\{n \in \mathbb{N} \mid$ *there exists an epimorphism of* $\mathbb{F}_\ell[G]$*-modules* $\varphi : \mathbb{F}_\ell[G]^n \to A\}$,

*and*

$\beta = \min\{m \in \mathbb{N} \mid$ *there exists a monomorphism of* $\mathbb{F}_\ell[G]$*-modules* $\Lambda : \mathfrak{X}(A) \to \mathbb{F}_\ell[G]^m\}$.

*Then* $\alpha = \beta$.

**Proof.** We have the following exact sequence

$$0 \longrightarrow \ker \varphi \longrightarrow \mathbb{F}_\ell[G]^\alpha \xrightarrow{\varphi} A \longrightarrow 0.$$

Applying Pontrjagin's dual and using (ii) and (iv) of Lemma 2.7, we obtain the exact sequence

$$0 \longrightarrow \mathfrak{X}(A) \longrightarrow \mathbb{F}_\ell[G]^\alpha \longrightarrow \mathfrak{X}(\ker \varphi) \longrightarrow 0.$$

Since $\beta$ is the minimum, it follows that $\beta \leq \alpha$. On the other hand, from the exact sequence

$$0 \longrightarrow \mathfrak{X}(A) \xrightarrow{\Lambda} \mathbb{F}_\ell[G]^\beta \longrightarrow \operatorname{coker}(\Lambda) \longrightarrow 0,$$

taking Pontrjagin's dual and using Lemma 2.7, we obtain

$$0 \longrightarrow \mathfrak{X}(\operatorname{coker}(\Lambda)) \longrightarrow \mathbb{F}_\ell[G]^\beta \longrightarrow A \longrightarrow 0.$$

Since $\alpha$ is the minimum, we have $\alpha \leq \beta$. □

**Proposition 4.2.** *With the notation as above, we have*

$$\dim_{\mathbb{F}_\ell}(\mathfrak{X}({}_\ell S)^G) = r, \tag{19}$$

*where r denotes the number of relations of the group G.*

**Proof.** Using the duality theorem for cohomology groups, that is, $H^{-j}(G, \mathfrak{X}(B)) \cong \mathfrak{X}(H^{j-1}(G, B))$, for all $j \in \mathbb{Z}$ (see [21], 4-4-6), we have

$$H^0(G, \mathfrak{X}({}_\ell S)) \cong \mathfrak{X}(H^{-1}(G, {}_\ell S)).$$

From (9), (14), and (4), we obtain $H^{-1}(G, {}_\ell S) \cong C_\ell^r$. Hence

$$H^0(G, \mathfrak{X}({}_\ell S)) \cong \mathfrak{X}(H^{-1}(G, {}_\ell S)) \cong \mathfrak{X}(C_\ell^r),$$

and since $S$ has no injective components, using Theorem 2.11, we conclude that $\dim_{\mathbb{F}_\ell} N(\mathfrak{X}({}_\ell S)) = 0$. Then $\mathfrak{X}({}_\ell S)^G \cong \mathfrak{X}(C_\ell^r)$. Finally by Lemma 2.7, we obtain $r = \dim_{\mathbb{F}_\ell}(\mathfrak{X}({}_\ell S)^G)$. □

Next, we present a different proof of Proposition 4.2, without using duality theorem for cohomology groups.

Taking $\ell$-parts in the exact sequence (15), we obtain the exact sequence

$$0 \to {}_\ell S \to \mathbb{F}_\ell[G]^d \to {}_\ell I_G \to 0. \tag{20}$$

Applying Pontrjagin's dual and using Lemma 2.7, we obtain the exact sequence

$$0 \to \mathfrak{X}({}_\ell I_G) \to \mathfrak{X}(\mathbb{F}_\ell[G]^d) \to \mathfrak{X}({}_\ell S) \to 0,$$

that is,

$$0 \to \mathfrak{X}({}_{\ell}I_G) \to \mathbb{F}_{\ell}[G]^d \to \mathfrak{X}({}_{\ell}S) \to 0. \tag{21}$$

On the other hand, we have the exact sequence

$$0 \to {}_{\ell}I_G \to \mathbb{F}_{\ell}[G] \to \mathbb{F}_{\ell} \to 0, \tag{22}$$

applying Pontrjagin's dual to (22) and using Lemma 2.7, we obtain the exact sequence

$$0 \to \mathbb{F}_{\ell} \to \mathbb{F}_{\ell}[G] \to \mathfrak{X}({}_{\ell}I_G) \to 0. \tag{23}$$

Therefore,

$$\mathfrak{X}({}_{\ell}I_G) \cong \frac{\mathbb{F}_{\ell}[G]}{\mathbb{F}_{\ell}} \text{ as } \mathbb{F}_{\ell}[G]\text{-modules.} \tag{24}$$

Now, from (24) and (21), we obtain the exact sequence

$$0 \to \frac{\mathbb{F}_{\ell}[G]}{\mathbb{F}_{\ell}} \to \mathbb{F}_{\ell}[G]^d \to \mathfrak{X}({}_{\ell}S) \to 0. \tag{25}$$

Since $\mathbb{F}_{\ell}[G]^d$ is cohomologically trivial, we obtain

$$\mathfrak{X}({}_{\ell}S)^G \cong H^1\left(G, \frac{\mathbb{F}_{\ell}[G]}{\mathbb{F}_{\ell}}\right).$$

From the exact sequences (23), (24), and since $\mathbb{F}_{\ell}[G]$ is cohomologically trivial, we have

$$H^1\left(G, \frac{\mathbb{F}_{\ell}[G]}{\mathbb{F}_{\ell}}\right) \cong H^2(G, \mathbb{F}_{\ell}).$$

On the other hand, from (8), we obtain

$$H^{\dot{2}}(G, \mathbb{F}_{\ell}) \cong H^2(G, {}_{\ell}R) \cong C_{\ell}^{\alpha_1(R)+\alpha_2(R)}. \tag{26}$$

Furthermore, we have the exact sequence

$$0 \to \mathbb{Z}_{\ell} \to \mathbb{Q}_{\ell} \to R \to 0.$$

Since $\mathbb{Q}_\ell$ is cohomologically trivial, we obtain

$$H^i(G, R) \cong H^{i+1}(G, \mathbb{Z}_\ell) \cong H^{i+1}(G, \mathbb{Z}). \tag{27}$$

Finally, from (26), (27), and (14), we have

$$\mathfrak{X}({}_\ell S)^G \cong H^2(G, \mathbb{F}_\ell) \cong C_\ell^{\alpha_2(\mathbb{Z})+\alpha_3(\mathbb{Z})} \cong C_\ell^{d+(r-d)} \cong C_\ell^r.$$

**Theorem 4.3.** *Let $L / K$ be a finite Galois $\ell$-extension of function fields and let $G = \mathrm{Gal}(L / K)$. Then*

(i) *There exists an $\mathbb{F}_\ell[G]$-epimorphism*

$$f : \mathbb{F}_\ell[G]^r \to {}_\ell S,$$

*where r is the number of relations of G.*

(ii) ${}_\ell S$ *is an indecomposable $\mathbb{F}_\ell[G]$-module.*

**Proof.** The proof of (i) is obtained from Lemma 3.3 and Propositions 4.2 and 4.1. Now, we assume that ${}_\ell S \cong A \oplus B$ for some non trivial $\mathbb{F}_\ell[G]$-modules $A$ and $B$, that is, ${}_\ell S$ is not indecomposable. From (9) and (4), we have

$$H^i(G, {}_\ell S) \cong C_\ell^{\alpha_{i-1}(\mathbb{Z})+\alpha_{i-2}(\mathbb{Z})}.$$

In particular, $H^1(G, {}_\ell S) \cong C_\ell^{\alpha_0(\mathbb{Z})+\alpha_{-1}(\mathbb{Z})} \cong C_\ell$. Then

$$C_\ell \cong H^1(G, {}_\ell S) \cong H^1(G, A) \bigoplus H^1(G, B),$$

thus $H^1(G, A) \cong \{0\}$ or $H^1(G, B) \cong \{0\}$. From [11], Theorem 5, page 142, we obtain $A \cong \mathbb{F}_\ell[G]^\alpha$ or $B \cong \mathbb{F}_\ell[G]^\beta$, which is impossible. Therefore, ${}_\ell S$ is an indecomposable $\mathbb{F}_\ell[G]$-module. □

Since ${}_\ell S$ is an $\mathbb{F}_\ell[G]$-module, $\mathbb{F}_\ell[G]^r$ is a projective $\mathbb{F}_\ell[G]$-module and $f : \mathbb{F}_\ell[G]^r \to {}_\ell S$ is an $\mathbb{F}_\ell[G]$-epimorphism, we may write $N = \ker f \cong N^{(0)} \oplus N^{(1)}$, where $N^{(1)}$ is a projective $\mathbb{F}_\ell[G]$-module and $N^{(0)}$ does

not contain any projective component, we have $\Omega({}_\ell S) = N^{(0)}$. By the Krull-Schmidt-Azumaya theorem, we have

$$\ker f \cong \Omega({}_\ell S) \bigoplus \mathbb{F}_\ell[G]^x. \tag{28}$$

From Theorem 4.3, we obtain the exact sequence of $\mathbb{F}_\ell[G]$-modules

$$0 \to \ker f \to \mathbb{F}_\ell[G]^r \to {}_\ell S \to 0. \tag{29}$$

Since $\mathbb{F}_\ell[G]^r$ is cohomologically trivial, we obtain

$$H^i(G, {}_\ell S) \cong H^{i+1}(G, \ker f). \tag{30}$$

**Proposition 4.4.** *We have what* $H^0(G, \ker f) \cong C_\ell^r$.

**Proof.** The result follows from (30), (9), (4), and (14). □

**Proposition 4.5.** *With the notation as above, we have*

(i) $\ker f \cong \Omega({}_\ell S)$.

(ii) *The following sequence of* $\mathbb{F}_\ell[G]$*-modules is exact*

$$0 \to \Omega({}_\ell S) \to \mathbb{F}_\ell[G]^r \to {}_\ell S \to 0. \tag{31}$$

**Proof.** (ii) follows from (i) and the exact sequence (29).

On the other hand, from (29), we obtain the exact sequence of cohomology groups

$$0 \to (\ker f)^G \to (\mathbb{F}_\ell[G]^r)^G \to {}_\ell S^G \to H^1(G, \ker f) \to 0.$$

Since $(\mathbb{F}_\ell[G]^r)^G = \mathbb{F}_\ell^r$, we have

$$\frac{|(\mathbb{F}_\ell)^r|}{|(\ker f)^G|} = \frac{|{}_\ell S^G|}{|H^1(G, \ker f)|}.$$

From (30), we obtain $H^1(G, \ker f) \cong H^0(G, {}_\ell S) \cong \dfrac{{}_\ell S^G}{N({}_\ell S)}$. Thus,

$$|(\ker f)^G| = \frac{|\mathbb{F}_\ell^r| \dfrac{|{}_\ell S^G|}{|N({}_\ell S)|}}{|{}_\ell S^G|} = \frac{|\mathbb{F}_\ell^r|}{|N({}_\ell S)|}.$$

Since $S$ has not components $R[G]$, applying Theorem 2.11, we have $N({}_\ell S) = \{0\}$, that is, $|N({}_\ell S)| = 1$. Therefore,

$$|(\ker f)^G| = |\mathbb{F}_\ell^r| = \ell^r.$$

Using Proposition 4.4, we obtain $|N(\ker f)| = 1$. Thus,

$$\dim_{\mathbb{F}_\ell} N(\ker f) = 0.$$

Finally, from Theorem 2.11, we obtain that $x = 0$ in (28), proving (i). □

**Proposition 4.6.** *With the notation as above, we have*

$${}_\ell S \cong \frac{\mathbb{F}_\ell[G]^r}{\Omega({}_\ell S)}.$$

**Proof.** By Proposition 4.5, we have the exact sequence of $\mathbb{F}_\ell[G]$-modules

$$0 \to \Omega({}_\ell S) \to \mathbb{F}_\ell[G]^r \to {}_\ell S \to 0. \tag{32}$$

On the other hand, we have the exact sequence of $\mathbb{F}_\ell[G]$-modules

$$0 \to \Omega({}_\ell S) \to \mathbb{F}_\ell[G]^r \to \frac{\mathbb{F}_\ell[G]^r}{\Omega({}_\ell S)} \to 0. \tag{33}$$

Applying, Schanuel's lemma (see Remark 2.3) to the exact sequences (32) and (33), we obtain the result. □

**Proposition 4.7.** *With the notation as above, we obtain*

$$\Omega^{\#}(\Omega({}_\ell S)) \cong \frac{\mathbb{F}_\ell[G]^r}{\Omega({}_\ell S)}.$$

**Proof.** The result is obtained by the exact sequence (33), and from the fact that $\dfrac{\mathbb{F}_\ell[G]^r}{\Omega({}_\ell S)}$ has no $\mathbb{F}_\ell[G]$-injective components. □

## 5. Implicit Characterization of ${}_\ell\mathscr{C}_{0L}$

The goals in this section are two. The first is to obtain an exact sequence of $\mathbb{F}_\ell[G]$-modules, which characterizes implicitly in all cases, the $\mathbb{F}_\ell[G]$-module structure of ${}_\ell\mathscr{C}_{0L}$. That is, we obtain implicitly the Galois module structure of ${}_\ell\mathscr{C}_{0L}$, this is, (39). The second objective is to obtain explicitly the Galois module structure of the dual of the $\ell$-part of the generalized Jacobian $\mathfrak{X}({}_\ell\mathbb{J}_{\mathfrak{N}})$.

We begin analyzing the decomposition as direct sum of indecomposable $\mathbb{Z}_\ell[G]$-modules of $\mathscr{C}_{0L}(\ell)$ for the case, when $L/K$ is unramified. In this case, the result is known in complete generality.

**Theorem 5.1** ([10], Rzedowski-Villa)**.** *Let $L/K$ be a finite Galois unramified $\ell$-extension of function fields with Galois group G. Then, the Galois module structure of $\mathscr{C}_{0L}(\ell)$ is given by*

$$\mathscr{C}_{0L}(\ell) \cong R[G]^{2(g_K-d)} \bigoplus \Omega^{\#}\left(\frac{R[G]}{R}\right) \bigoplus S, \tag{34}$$

*where* $\Omega^{\#}\left(\dfrac{R[G]}{R}\right) \cong \dfrac{R[G]^d}{\dfrac{R[G]}{R}}$ *and $S$ is the $\mathbb{Z}_\ell[G]$-module given in the Theorem* 3.1.

**Proof.** See, [10], Theorem 3.1, page 558. □

In particular, considering the $\ell$-part of $\mathscr{C}_{0L}(\ell)$, from (34), we obtain

$${}_\ell\mathscr{C}_{0L} \cong \mathbb{F}_\ell[G]^{2(g_K-d)} \bigoplus \left(\frac{\mathbb{F}_\ell[G]^d}{\dfrac{\mathbb{F}_\ell[G]}{\mathbb{F}_\ell}}\right) \bigoplus {}_\ell S. \tag{35}$$

The following result exhibits a relationship between the $\mathbb{F}_\ell[G]$-modules ${}_\ell S$ and $\dfrac{\mathbb{F}_\ell[G]}{\mathbb{F}_\ell}$.

**Proposition 5.2.** *If* $T = \frac{\mathbb{F}_\ell[G]}{\mathbb{F}_\ell}$ *and* ${}_\ell S$ *is the* $\ell$*-part of the* $\mathbb{Z}_\ell[G]$*-module* $S$*, then*

$$ {}_\ell S \cong \mathfrak{X}(\Omega^{\#}(T)) \text{ as } \mathbb{F}_\ell[G]\text{-modules.} $$

**Proof.** From Proposition 3.7, we have $I_G \cong \Omega^{\#}(S)$. Taking $\ell$-parts, we obtain ${}_\ell I_G \cong \Omega^{\#}({}_\ell S)$. From Pontrjagin's dual, we obtain $\mathfrak{X}(I_G) \cong \mathfrak{X}(\Omega^{\#}({}_\ell S))$. Lemma 2.7, (iii), implies that $\mathfrak{X}({}_\ell I_G) \cong \frac{\mathbb{F}_\ell[G]}{\mathbb{F}_\ell}$. Therefore,

$$ \frac{\mathbb{F}_\ell[G]}{\mathbb{F}_\ell} \cong \mathfrak{X}(\Omega^{\#}({}_\ell S)). $$

Using properties of $\mathfrak{X}$ and $\Omega$ (see Proposition 2.9), we obtain

$$ \frac{\mathbb{F}_\ell[G]}{\mathbb{F}_\ell} \cong \mathfrak{X}(\Omega^{\#}({}_\ell S)) \cong \Omega(\mathfrak{X}({}_\ell S)). $$

Taking Pontrjagin's dual and using Proposition 2.4, we have

$$ \mathfrak{X}(T) \cong \mathfrak{X}(\Omega(\mathfrak{X}({}_\ell S))) \cong \mathfrak{X}(\mathfrak{X}(\Omega^{\#}({}_\ell S))) \cong \Omega^{\#}({}_\ell S). $$

Applying Heller's loop operator, we obtain

$$ \Omega(\mathfrak{X}(T)) \cong \Omega(\Omega^{\#}({}_\ell S)) \cong {}_\ell S. $$

Finally, we have

$$ \mathfrak{X}(\Omega^{\#}(T)) \cong {}_\ell S. $$ □

In the unramified case, we obtain from Proposition 5.2 and (35)

$$ {}_\ell \mathscr{C}_{0L} \cong \mathbb{F}_\ell[G]^{2(g_K - d)} \bigoplus \left( \frac{\frac{\mathbb{F}_\ell[G]^d}{\mathbb{F}_\ell[G]}}{\mathbb{F}_\ell} \right) \bigoplus \mathfrak{X}\left( \frac{\frac{\mathbb{F}_\ell[G]^d}{\mathbb{F}_\ell[G]}}{\mathbb{F}_\ell} \right), \tag{36} $$

where $g_K$ denotes the genus of $K$ and $d$ denotes the minimum number of generators of $G$.

On the other hand, applying Pontrjagin's dual to ${}_\ell\mathbb{J}_{\mathfrak{N}}$ in (5), we obtain

$$\mathfrak{X}({}_\ell\mathbb{J}_{\mathfrak{N}}) \cong \mathfrak{X}\Big(\mathbb{F}_\ell[G]^{2g_K+t-1-d} \bigoplus {}_\ell S\Big). \tag{37}$$

Since, for $\mathbb{F}_\ell[G]$-modules $M_1$ and $M_2$, we have

$$\mathfrak{X}(M_1 \oplus M_2) = \mathrm{Hom}_{\mathbb{F}_\ell}(M_1 \oplus M_2, \mathbb{F}_\ell) \cong \mathrm{Hom}_{\mathbb{F}_\ell}(M_1, \mathbb{F}_\ell) \oplus \mathrm{Hom}_{\mathbb{F}_\ell}(M_2, \mathbb{F}_\ell)$$

$$= \mathfrak{X}(M_1) \oplus \mathfrak{X}(M_2).$$

If follows that

$$\mathfrak{X}({}_\ell\mathbb{J}_{\mathfrak{N}}) \cong \mathfrak{X}(\mathbb{F}_\ell[G]^{2g_K+t-1-d}) \bigoplus \mathfrak{X}({}_\ell S).$$

We have proved, using Lemma 2.7 and Propositions 2.4 and 5.2.

**Theorem 5.3.** *Let $L/K$ be a finite Galois $\ell$-extension of function fields with field of constants $k$, an algebraically closed of characteristic $p \neq \ell$. Then, the $\mathbb{F}_\ell[G]$-module structure of the dual of the $\ell$-part of the generalized Jacobian $\mathfrak{X}({}_\ell\mathbb{J}_{\mathfrak{N}})$ is given by*

$$\mathfrak{X}({}_\ell\mathbb{J}_{\mathfrak{N}}) \cong \mathbb{F}_\ell[G]^{2g_K+t-1-d} \bigoplus \frac{\mathbb{F}_\ell[G]^d}{\frac{\mathbb{F}_\ell[G]}{\mathbb{F}_\ell}}. \qquad \square$$

Next, we are interested in analyzing the $\mathbb{F}_\ell[G]$-module structure of the $\ell$-part of the usual Jacobian $\mathscr{C}_{0L}(\ell)$ in the general case, i.e., the implicit characterization of ${}_\ell\mathscr{C}_{0L}$ for an arbitrary finite $\ell$-extension.

Taking $\ell$-parts in the exact sequence (1), we obtain the exact sequence of $\mathbb{F}_\ell[G]$-modules

$$0 \longrightarrow {}_\ell\mathfrak{R} \longrightarrow {}_\ell\mathscr{C}_{0\mathfrak{N}} \xrightarrow{\pi} {}_\ell\mathscr{C}_{0L} \longrightarrow 0. \tag{38}$$

From Theorems 3.1 and 4.3, we have

$$\begin{array}{ccc} {}_\ell\mathscr{C}_{0\mathfrak{N}} & \cong \mathbb{F}_\ell[G]^{2g_K-1+t-d}\oplus & {}_\ell S \\ & \uparrow \mathrm{id} & \uparrow f \\ \mathbb{F}_\ell[G]^{2g_K-1+t-d+r} & \cong \mathbb{F}_\ell[G]^{2g_K-1+t-d}\oplus & \mathbb{F}_\ell[G]^r \end{array}$$

That is, we have an $\mathbb{F}_\ell[G]$-epimorphism

$$f_1 := (\mathrm{id},\, f) : \mathbb{F}_\ell[G]^{2g_K+t-1-d+r} \to {}_\ell\mathscr{C}_{0\mathfrak{N}}.$$

Then,

$$\Phi = \pi \circ f_1 : \mathbb{F}_\ell[G]^{2g_K+t-1-d+r} \to {}_\ell\mathscr{C}_{0L}$$

is an $\mathbb{F}_\ell[G]$-epimorphism.

Therefore, we have the commutative diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & {}_\ell\mathfrak{R} & \longrightarrow & {}_\ell\mathscr{C}_{0\mathfrak{N}} & \xrightarrow{\pi} & {}_\ell\mathscr{C}_{0L} & \longrightarrow & 0 \\ & & \uparrow & & \uparrow f & \nearrow_{\Phi} & & & \\ 0 & \longrightarrow & \ker\Phi & \longrightarrow & \mathbb{F}_\ell[G]^{2g_K-1+t-d+r} & & & & \end{array}$$

From Schanuel's lemma (see Remark 2.3) follows that the diagram characterizes ${}_\ell\mathscr{C}_{0L}$ as $\mathbb{F}_\ell[G]$-module. In short, we obtain the exact sequence of $\mathbb{F}_\ell[G]$-modules

$$0 \longrightarrow \ker\Phi \longrightarrow \mathbb{F}_\ell[G]^{2g_K-1+t-d+r} \xrightarrow{\Phi} {}_\ell\mathscr{C}_{0L} \longrightarrow 0. \quad (39)$$

**Theorem 5.4.** *Let $L/K$ be a finite abelian $\ell$-extension of function fields with field of constants $k$, an algebraically closed of characteristic $p \neq \ell$. Then, the $\mathbb{F}_\ell[G]$-module structure of ${}_\ell\mathscr{C}_{0L}$ is given implicitly by*

$${}_\ell\mathscr{C}_{0L} \cong \mathbb{F}_\ell[G]^{2(g_K-d_0)} \bigoplus \Omega^{\#}(\ker\Phi), \quad (40)$$

*where $d_0$ denotes the minimum number of generators of $G/T = \mathrm{Gal}(K^{nr}/K)$, with $K^{nr}$ the maximal unramified extension of $K$ in $L$.*

**Proof.** By the Krull-Schmidt-Azumaya theorem (see [1]), we have

$${}_\ell\mathscr{C}_{0L} \cong A \bigoplus B,$$

where $A$ is an injective $\mathbb{F}_\ell[G]$-module and $B$ has no injective components. Applying the dual of Heller's loop operator to the exact sequence (39), we obtain

$$B \cong \Omega^{\#}(\ker \Phi).$$

On the other hand, for $A$, in [4], it was proved, in the case, when $L/K$ is a finite abelian $\ell$-extension, what $A \cong \mathbb{F}_\ell[G]^{2(g_K - d_0)}$, where $d_0$ denotes the minimum number of generators of $G/T = \mathrm{Gal}(K^{nr}/K)$, with $K^{nr}$, the maximal unramified extension of $K$ in $L$. □

## References


[1] C. W. Curtis and K. R. Reiner, Methods of Representation Theory with Applications to Finite Groups and Orders, Vol. 1, John Wiley & Sons, New York, 1981.

[2] P. J. Hilton and U. Stammbach, A Course in Homological Algebra, Graduate Texts in Mathematics 4, Berlin-Heidelberg-New York, 1971.

[3] F. Jarquín-Zárate and G. Villa-Salvador, Cohomología y Jacobianos, Aportaciones Matemáticas 37 (2007), 67-84.

[4] F. Jarquín-Zárate and G. Villa-Salvador, Integral representation of generalized Jacobians, International Journal of Algebra 3(16) (2009), 747-774.

[5] G. Karpilovsky, Group Representations, North-Holland Mathematics Studies, 175 Vol. I, North-Holland, Amsterdam-London-New York-Tokyo, 1992.

[6] P. R. López-Bautista and G. D. Villa-Salvador, Integral representation of $p$-class groups in $\mathbb{Z}_p$-extensions and the Jacobian variety, Can. J. Math. 50 (1998), 1253-1272.

[7] V. J. Mejía-Huguet and M. Rzedowski-Calderón, Galois modular representation of associated Jacobians in the Tamely ramified cyclic case, Manuscripta Math. 126 (2008), 531-543.

[8] S. Morris, Pontryagin Duality and the Structure of Locally Compact Abelian Groups, Cambridge University Press, Cambridge, 1977.

[9] M. Rzedowski-Calderón, G. D. Villa-Salvador and M. L. Madan, Galois module structure of Tate modules, Math. Z. 224 (1997), 77-101.

[10] M. Rzedowski-Calderón and G. D. Villa-Salvador, Galois module structure of Jacobians in unramified extensions, Journal of Algebra 242 (2001), 550-560.

[11] J. P. Serre, Local Fields, Graduate Texts in Mathematics 67, Springer-Verlag, Berlin/Heidelberg/New York, 1979.

[12] J. P. Serre, Algebraic Groups and Class Fields, Graduate Texts in Mathematics 117 Berlin/Heidelberg/New York, 1988.

[13] J. P. Serre, Galois Cohomology, Springer-Verlag, Berlin-Heidelberg, 1997.

[14] R. Valentini, Representations of automorphisms on differentials of function fields of characteristic $p$, J. Reine und Angew. Math. 335 (1982), 164-179.

[15] R. Valentini, Some $p$-adic Galois representations for curves in characteristic $p$, Math. Z. 192 (1986), 541-545.

[16] G. D. Villa-Salvador and M. L. Madan, Structure of semisimple differentials and $p$-class groups in $\mathbb{Z}_p$-extensions, Manuscripta Math. 57 (1987), 315-350.

[17] G. D. Villa-Salvador and M. L. Madan, Integral representations of $p$-class groups in $\mathbb{Z}_p$-extensions, semisimple differentials, and Jacobians, Arch. Math. 56 (1991), 254-269.

[18] G. D. Villa-Salvador and M. Rzedowski-Calderón, Galois module structure of generalized Jacobians, Revista Matemática de la Universidad Complutense de Madrid 10 (1997), 39-51.

[19] G. D. Villa-Salvador, Introducción a la Teoría de las Funciones Algebraicas, Fondo de Cultura Económica, México, 2003.

[20] G. D. Villa-Salvador, Topics in the Theory of Algebraic Function Fields, Birkhäuser, Boston-Basel-Berlin, 2006.

[21] E. Weiss, Cohomology of Groups, Academic Press, New York/London, 1969.

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